This month’s column, the second in a series of three on compressible flow, derives the key relationships for understanding steady, isentropic flow in a converging-diverging nozzle. The starting point of these derivations are the three isentropic jump conditions derived in the last post:

\[ \begin{array}{ll} {\textrm{continuity:}}& {\rho}\_1 V\_1 A\_1 = {\rho}\_2 V\_2 A\_2 \equiv {\dot m}\\{\textrm{momentum:}}& (\rho\_1 V\_1^2 + P\_1) A\_1 = (\rho\_2 V\_2^2 + P\_2) A\_2 \\{\textrm{energy:}}& h\_1 + \frac{{V\_1}^2}{2} = h\_2 + \frac{{V\_2}^2}{2} \end{array} \; , \]

along with the first law of thermodynamics (cast in intensive variables)

\[ de = T ds + \frac{P}{\rho^2} d \rho \; ,\]

the equation of state for an ideal gas

\[ P \rho = R T \; , \]

and the fact that the specific internal energy and specific enthalpy are simply functions of temperature given by $$e = c\_{{\mathcal V}} T$$ and $$h = c\_P T$$, respectively. Other thermodynamic relations will be taken as given since proof will take us to far afield.

There are three ingredients in understanding the [steady, isentropic flow through a converging-diverging nozzle](https://en.wikipedia.org/wiki/Isentropic_nozzle_flow) (also known as a [de Laval nozzle](https://en.wikipedia.org/wiki/De_Laval_nozzle)): 1) calculating the Mach number, 2) relating the thermodynamics variables of temperature ($$T$$), the pressure ($$P$$) and the density ($$\rho$$) to the Mach number, and 3) relating the cross-sectional area of the nozzle ($$A$$) to the Mach number.

This post is a synthesis and clarification of the ideas and materials found in Chapter 9 of Merle Potter’s *Fluid Mechanics Demystified* and the 5-part series of YouTube lectures on compressible fluid flow from the University of Florida ([part 1](https://www.youtube.com/watch?v=CnAEwDQxeKo), [part 2](https://www.youtube.com/watch?v=ZzyU7BHCllY), [part 3](https://www.youtube.com/watch?v=o-UGasX1atc), [part 4](https://www.youtube.com/watch?v=MdYYgyHNDFY), and [part 5](https://www.youtube.com/watch?v=L-6gvVuiAQ4))

# Calculating the Mach Number

The central parameter in the theory is the Mach number, which is innocently defined as the ratio of the flow speed to the speed of sound. What makes this definition so deceptively simple looking is we are used to thinking that the speed of sound is constant since the temperature range that we normally encounter is such that the variations in the speed of sound is negligible. However, interesting compressible fluid flows often have very large variations in temperature and so Mach number varies depending on the location of the flow within the nozzle.

To calculate the speed of sound, which we will denote as $$c$$, we take the first differential of the continuity and momentum equations and since we are interested in the local speed we can ignore variations in the area. The resulting equations are:

\[ d \rho c + \rho d c = 0 \; \]

and

\[ d\rho c^2 + 2 c \rho dc + dP = 0 \; . \]

Solving the first equation for $$\rho dc = - c d \rho$$, substituting this result into the second, and solving for $$c^2$$ gives

\[ c^2 = \frac{d P}{d \rho} \; \]

An ideal gas obeys a polytropic equation of state

\[ P = \rho^{\gamma} \; \]

e first equation into the second and solving for